EEL 5542 Random Processes
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6. $R_X(\tau)$ is a non-negative definite function, i.e., for any function $f(t)$,

$$\int_a^b \int_a^b R(t,s) f(t) f^*(s)\, dt\, ds \geq 0$$

<u>proof</u> $E\left[\left| \int_a^b X(t) f(t)\, dt \right|^2 \right] \geq 0$

$$\Rightarrow E\left[\int_a^b \int_a^b X(t) f(t) X^*(s) f^*(s)\, dt\, ds \right] \geq 0$$

$$\Rightarrow \int_a^b \int_a^b E\left[X(t) X^*(s) \right] f(t) f^*(s)\, dt\, ds \geq 0$$

$$\Rightarrow \int_a^b \int_a^b R(t,s) f(t) f^*(s)\, dt\, ds \geq 0 \quad \blacksquare$$

Hence R is non-negative definite.

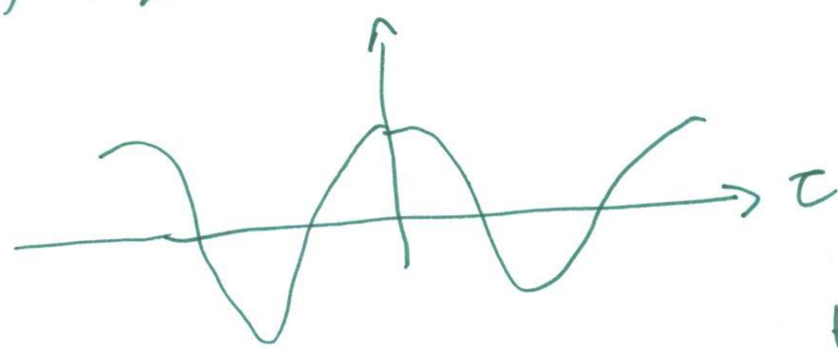
Bochner's Theorem

Any function $R(\tau)$ is non-negative definite if and only if its Fourier transform is non-negative.

Examples of Autocorrelation Functions

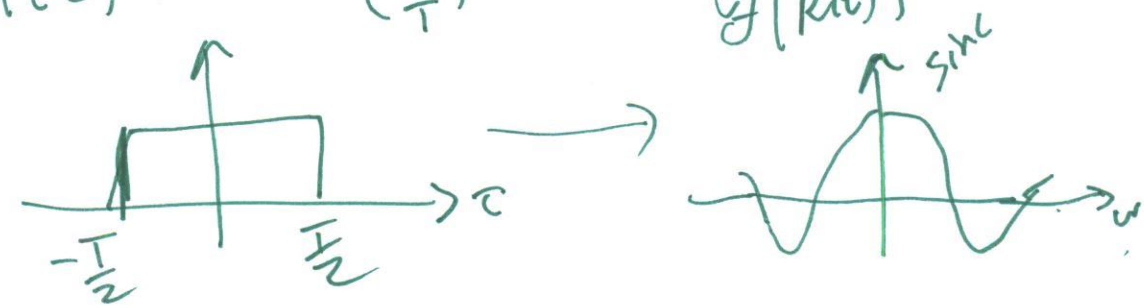
1.) $R_X(\tau) = e^{-2|\tau|}$

2.) $R_X(\tau) = a^2 \cos(2\pi f_0 \tau)$

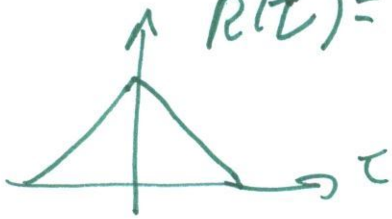


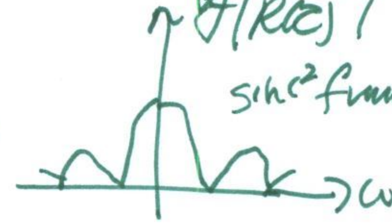
3.) $R(\tau) = \text{rect}\left(\frac{\tau}{T}\right)$ is <u>not</u> a autocorrelation function since its FT is not non-negative

$\mathcal{F}[R(t)]$



4.) $R(\tau)$ = triangle function





Random bit streams look like this

Power spectral Density

$$\mathbb{X}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \text{exists if} \int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

The power spectral density of $X(t)$ is

$$\phi_{XX}(\omega) = \mathcal{F}[R_{\mathbb{X}\mathbb{X}}(\tau)]$$

Properties

1.) $\phi_{\mathbb{X}\mathbb{X}}(\omega)$ is real and ≥ 0 for any ω

2.) $R_{\mathbb{X}}(\tau) = \mathcal{F}^{-1}[\phi_{XX}(\omega)]$

3.) $\phi_{XX}(\omega)$ is even if $X(t)$ is real

4.) $E[|X(t)|^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{XX}(\omega)\, d\omega$

Time-Invariant Linear Systems

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

for deterministic signals

$h(t)$ = impulse response

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

The mean of $Y(t)$ is

$$m_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\lambda)\, X(t-\lambda)\, d\lambda\right]$$

$$= \int_{-\infty}^{\infty} h(\lambda)\, E[X(t-\lambda)]\, d\lambda$$

$$= \int_{-\infty}^{\infty} h(\lambda)\, m_X(t-\lambda)\, d\lambda$$

$$= h(t) * m_X(t)$$

- The autocorrelation for WSS random process $X(t)$ becomes

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\lambda)\,h(\sigma)\,R_X(\tau-\lambda+\sigma)\,d\lambda\,d\sigma$$

The power spectral density becomes

$$\phi_{YY}(s) = H(s)\,H(-s)\,\phi_{XX}(s)$$

$$\Rightarrow \phi_{YY}(\omega) = H(\omega)\,H^*(\omega)\,\phi_{XX}(\omega)$$

$$\phi_{YY}(\omega) = |H(\omega)|^2\,\phi_{XX}(\omega)$$

Example

$$X(t) \longrightarrow \boxed{H(s)} \longrightarrow Y(t)$$

$$m_X(t) \longrightarrow m_Y(t) = h(t) * m_X(t)$$

$$\phi_{XX}(\omega) \longrightarrow \phi_{YY}(\omega) = |H(\omega)|^2\,\phi_{XX}(\omega)$$

Bold play vs conservative play
(gambling)