HW #1 Chap 1 # 5, 25, 32, 33, 45
(Due Wed)

The events A and B are
stat. indp. if \( P(AB) = P(A)P(B) \).

Applications

1. Randomized Sampling

   Consider a "sensitive survey"
   
   \[ A = \text{embarrassing event} \]
   \[ B = \text{nonsensitive random event, stat. indp. of } A \text{ and with known stat. properties } (P(B) \text{ is known}) \]
   
   \( A \cup B \) is a non-sensitive question
   (does not disclose event A).
\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{Law of Addition} \]

\[ = P(A) + P(B) - P(A)P(B) \quad \text{stat. indep.} \]

\[ \Rightarrow \text{from survey} = P(A)[1 - P(B)] + P(B) \quad \text{prob. properties of B} \]

\[ \Rightarrow P(A) = \frac{P(A \cup B) - P(B)}{1 - P(B)} \quad \text{stat. indep.} \]

Remarks (Reality Check)

1. The values here are between 0 and 1.
   (Why is this true for the above equation?)

2. The survey may give numbers that are not between 0 and 1. Need to have a big enough sample size and the subjects must be honest.
2.) Typos in a Book

Two proofreaders A and B proofread a long book.
A has a prob $P_A$ of catching a given typo
B has $P_B$

Assume these two probs are std. indep.

How do we estimate how many typos are not caught by either A or B?

Let $n_A = \#$ of typos caught by A
$n_B = \#$ of typos caught by B
$n_{AB} = \#$ of typos caught by both A and B
$n = \#$ of typos in the book (unknown)

Assume $n_A = nP_A$, $n_B = nP_B$, and
$n_{AB} = nP_{AB} = nP_A P_B$ are std. indep.
Note that this implies
\[ n_{AB} = \frac{n \cdot p_A \cdot p_B}{n} = \frac{n_A \cdot n_B}{n} \]

\[ \Rightarrow n = \frac{n_A \cdot n_B}{n_{AB}} \]

# of remaining typos = \[ n - [n_A + n_B - n_{AB}] \]

\[ = n - n_A - n_B + n_{AB} \]

\[ = \frac{n_A \cdot n_B}{n_{AB}} - n_A - n_B + n_{AB} \]

\[ = \frac{n_A \cdot n_B - n_A n_{AB} - n_B n_{AB} + n_{AB}^2}{n_{AB}} \]

# of remaining typos ≤ \[ \frac{(n_A - n_{AB})(n_B - n_{AB})}{n_{AB}} \]

We want \( n_A = n_{AB} \) or \( n_B = n_{AB} \) or something close to that. We do not want \( n_{AB} \approx 0 \).

This method is used in seismology.
3.) Jurist Problems

Suppose a group of jurists each has prob $p$ of making a correct decision independent of the others.

(a) Which jury is more likely to make a correct decision:

1 jurist or 3 jurists?

$p(\text{correct with one jurist}) = p$
$p(\text{correct with three jurists}) = p^3 + 3p^2q$ (where $q = 1-p$)

To see which is better, larger:

$p^3 + 3p^2q > p$
$p^3 + 3p^2(1-p) > p$
$p^3 + 3p^2 - 3p^3 > p$

We conclude that $p(\text{correct with 3 jurists}) > p(\text{correct with 1 jurist})$ if $\frac{1}{2} < p < 1$.
and \( P(\text{correct with 3 jurists}) < P(\text{correct with 1 jurist}) \)
if \( 0 < p < \frac{1}{2} \)

The two are equal when \( p = 0 \) or \( p = 1 \) or \( p = \frac{1}{2} \).

(5) (The Flippant Jurist) Same situation as above with 3 jurists but the 3rd jurist flips a fair coin to make a decision.

\[
P(\text{correct decision}) = p^2 \left(\frac{1}{2}\right) + p^2 \left(\frac{1}{2}\right) + 2pq \left(\frac{1}{2}\right) \\
= p^2 + pq \\
= p(p+q) \\
= p
\]

This example has implications to sensors.
The Law of Total Probability

def A collection $B_1, \ldots, B_n$ of subsets of $S$ forms a partition of $S$ if

$$\bigcup_{i=1}^{n} B_i = S \quad (B_1, \ldots, B_n \text{ are exhaustive})$$

and $B_i \cap B_j = \emptyset$ for $i \neq j$ ($B_i, \ldots, B_n$ are mutually exclusive)

Theorem (The Law of Total prob.)

Let $B_1, \ldots, B_n$ form a partition of $S$. Then

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$

Proof $P(A) = P(A \cap S) = P\left( A \cap \left( \bigcup_{i=1}^{n} B_i \right) \right)$

$$= P\left( \bigcup_{i=1}^{n} (A \cap B_i) \right)$$

$$= \sum_{i=1}^{n} P(A \cap B_i) \text{ since } (A \cap B_i) \cap (A \cap B_j) = \emptyset$$

for $i \neq j$ (i.e., Axiom (A5)).
\[
P(A) = \sum_{i=1}^{n} \frac{P(A \cap B_i)}{P(B_i)} P(B_i)
\]

\[
= \sum_{i=1}^{n} \frac{1}{2} P(A \mid B_i) P(B_i)
\]

This approach is called conditioning.

Example: Consider a fair deck of cards that is well shuffled. Two cards are drawn in order. (a) What is the prob that the 1st card is the ace of spades?

\[\text{Ans: } \frac{1}{52}\]

(b) What is the prob that the 2nd card is the ace of spades?

\[\text{Ans: } P(2\text{nd card is ace of spades})
= P(2\text{nd card is } \text{AS} \mid \text{1st card} = \text{AS}) P(1\text{st card} = \text{AS}) + \]
\[ P(\text{2nd card is } \underline{\text{AS}}) = P(\text{1st card is not AS}) \]

\[ = 0 \left( \frac{1}{52} \right) + \frac{1}{51} \left( \frac{51}{52} \right) \]

\[ = \frac{1}{52} \quad \text{as we should.} \]

**Bayes' Theorem**

Let \( A_1, \ldots, A_n \) be a partition of \( S \). Then

\[ P(A_j|B) = \frac{P(B|A_j) P(A_j)}{\sum_{i=1}^{n} P(B|A_i) P(A_i)} \]