Chapter 10. Moments of Inertia

Lecture 1/2

EGM 3512: Engineering Mechanics
Instructor: Sungmoon Jung (sjung@eng.fsu.edu)
Why Is the I-Beam Oriented This Way, Not the Other Way?

→ Answer: Higher Moment of Inertia
What We Can Do After Learning This Chapter

• What are the moment of inertia of the I-beams?
• What size of I-beams are safe to use in this building? (EGN 3331, Strength of Materials)

Image from: http://www.gtgrandstands.com/grandstand.htm
What We Need to Learn

• **Definition of the moment of inertia**
• Parallel axis theorem
• To calculate the moment of inertia
  ▫ Single area using integration
  ▫ Composite areas using table
• Mass moment of inertia
What We Need to Learn

• Definition of the moment of inertia
• Parallel axis theorem
• To calculate the moment of inertia
  ▫ Single area using integration
  ▫ Composite areas using table
• Mass moment of inertia
Definition

- Recall: centroid of area

\[ \bar{x} = \frac{\int x \, dA}{\int dA} \]

- Moment of inertia: second moment of the area

\[ \text{moment of inertia} = \int y^2 \, (dA) \]

(first) moment of the force: \( y \, (dF) \)

(first) moment of the area: \( y \, (dA) \)

(second " " " " : \( y^2 \, (dA) \)
Definition (Continued)

\[ I_x = \int y^2 \, dA \]
\[ I_y = \int x^2 \, dA \]

polar moment of inertia (about the "pole" 0)

\[ I_0 = \int r^2 \, dA = I_x + I_y \]
Definition (Continued)

• Radius of Gyration

\[ k_x = \sqrt{\frac{I_x}{A}} \]

\[ k_y = \sqrt{\frac{I_y}{A}} \]

\[ k = \sqrt{\frac{I_o}{A}} \]
Moment of Inertia in Structural Analysis

- Given the same magnitude of the load, which one will deflect more? How do we quantify the deflection?

\[ F = ku \]
\[ F = \frac{3EI}{L^3} u \]
Moment of Inertia in Structural Analysis (Continued)

- Stresses in beams (from EGN3331: Strength of Materials)

\[ \sigma = \frac{My}{I} \]
Moment of Inertia in Structural Analysis (Continued)

- Torsion (from EGN3331: Strength of Materials)

\[ T = \frac{G I}{L} \phi \]
Quiz

• The definition of the moment of inertia involves an integral of the form

\[ \int x^2 \, dA \]

- \( \int x \, dA \) ×
- \( \int x^2 \, dA \)
- \( \int x^2 \, dm \)
- \( \int m \, dA \)

• Select the SI units for the moment of inertia

- \( m^3 \)
- \( m^4 \)
- \( \text{kg} \, m^2 \)
- \( \text{N} \, m^2 \)
Example 1A

- Compute the moment of the inertia about x-axis

\[ I_x = \int y^2 \, dA \]
\[ = \int_0^2 y^2 \left(2 - x\right) \, dy \]
\[ = \int_0^2 y^2 \left(2 - \frac{y^2}{2}\right) \, dy \]
\[ = \left[ \frac{2}{3} y^3 - \frac{1}{10} y^5 \right]_0 \]
\[ = 2.133 \, m^4 \]
Example 1B

- Compute the moment of the inertia about y-axis

\[
I_y = \int y^2 \, dA
\]

\[
= \int_0^2 x^2 \cdot y \, dx
\]

\[
= \int_0^2 x^2 \sqrt{2x} \, dx
\]

\[
= \left[ \sqrt{x} \cdot \frac{x^{\frac{7}{2}}}{7} \right]_0^2
\]

\[
= 4.571 \text{ m}^4
\]
What We Need to Learn

- Definition of the moment of inertia
- Parallel axis theorem
- To calculate the moment of inertia
  - Single area using integration
  - Composite areas using table
- Mass moment of inertia
Motivation for Parallel-Axis Theorem

- Given $I_x$ (or, after computing $I_x$), we often have to compute the moment of inertia for a new axis
  - Rather than new integration, a theorem would be useful
- Many structural members are composed of simple shapes, in which we have “table” for computing them
  - A theorem would be useful, that enables us to combine moments of inertia of all simple shapes
Parallel-Axis Theorem

Given: \( C, I_{x'}, I_{y'}, I_C \);  
Find: \( I_x, I_y, I_O \)

\[
I_x = \int y^2 \, dA = \int (d_x + y')^2 \, dA \\
= \int d_x^2 \, dA + \int 2d_x y' \, dA + \int (y')^2 \, dA \\
= d_x^2 A + 0 + I_{x'} \\
(Since \( C \) is centroid, \( y' = \frac{\int y' \, dA}{\int dA} = 0 \))
\]

\[
I_x = I_{x'} + Ad_x^2 \\
I_y = I_{y'} + Ad_y^2 \\
I_O = I_C + Ad^2
\]
Example 2

• Compute moment of inertia about $x'$ axis, $x_b$ axis, and $C$.

$$I_{x'} = \int (y')^2 \, dA = \int (y')^2 \, b \, dy'$$

$$= \left[ \frac{b}{3} (y')^3 \right]_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{1}{12} bh^3$$

$$I_{x_b} = I_{x'} + A \, dy^2$$

$$= \frac{1}{12} bh^3 + (bh) (\frac{h}{2})^2 = \frac{bh^3}{3}$$

$$I_{y'} = \frac{1}{12} h b^3$$ (by observation $\text{switch } x_b \leftrightarrow y_b$)

$$\chi_b \quad I_C = I_{x'} + I_{y'} = \frac{1}{12} bh (h^2 + b^3)$$
Example 2 (Continued)

• The above example illustrates how we can derive equations for moments of inertia for simple shapes
• If you go to the back of the textbook:

\[ I_x = \frac{1}{12} bh^3 \]
\[ I_y = \frac{1}{12} hb^3 \]

• All other equations can be obtained similarly
Example 3

- Compute $I_x$

\[
I_x = \int y^2 \, dA
\]

\[
= \int_0^{200} y^2 \left(100 - x\right) \, dy
\]

\[
= \int_0^{200} y^2 \left(100 - \frac{y^4}{400}\right) \, dy
\]

\[
= \left[ \frac{100y^3}{3} - \frac{1}{2000} y^5 \right]_0^{200}
\]

\[
= 106.7 \times 10^6 \text{ mm}^4
\]
Example 3 (Alternative Approach)

What if we take a vertical slice? → (moment arm) \(2 \times \) (area), but the moment arm changes over the height.

Method 1: double integral

\[
\int_0^{100} (\text{arm})^2 dA = \int_0^{100} \left( \int_0^{y'/2} (\frac{y}{2})^2 dy' \right) dx
\]

\[
= \int_0^{100} \left[ \frac{1}{3} \left( \frac{y^3}{2} \right) \right]_0^{100} dx = \int_0^{100} \left( \frac{1}{3} \cdot \frac{y^3}{2} \right) dx
\]

\[
= \int_0^{100} \frac{1}{3} \left( 400x \right)^{\frac{3}{2}} dx
\]

\[
= \left[ \frac{1}{3} \cdot \frac{2}{5} \left( 400x \right)^{\frac{5}{2}} \cdot \frac{1}{400} \right]_0^{100}
\]

\[
= 106.7 \times 10^6 \text{ mm}^4
\]
Example 3 (Alternative App, Cont’d)

Method 2: use table and the parallel axis theorem

\[ \int_0^{100} dI_x = \int_0^{100} \left( \frac{1}{12} (dx) y^3 + (dx)(y)(\frac{y^2}{2}) \right) = \int_0^{100} \left( \frac{1}{3} y^3 \right) dx \]

\[ = 106.7 \times 10^6 \text{ mm}^4 \]
Example 3 (Summary)

- Unlike the centroid, it is always easiest to take the “slice” parallel to the axis that it is rotating about.
- If you have to take the “slice” vertical to the axis, you have to use:
  - Double integral, or
  - Table (equation for simple shape) with the parallel axis theorem.
What We Learned in This Lecture

- Definition of the moment of inertia
- Parallel axis theorem
- To calculate the moment of inertia
  - Single area using integration
  - Composite areas using table
- Mass moment of inertia

The class has not finished yet. Review the PowerPoint slides, then take the quiz in the Blackboard.