M - 11/27  Exam III this Wednesday

Chap 9 Global Nonlinear Techniques

\[ \dot{x} = f(x) \]
\[ \dot{x} = f(x,y) \]
\[ \dot{y} = g(x,y) \]

\textit{x-nullclines} - where \( \dot{x} = 0 \)
\textit{y-nullclines} - where \( \dot{y} = 0 \)

Example
\[ \dot{x} = y - x^2 \]
\[ \dot{y} = x - 2 \]

\textit{x-nullcline} \( y = x^2 \)
\textit{y-nullcline} \( x = 2 \)

\( x > 0 \) above parabola
\( x < 0 \) below parabola
\( y > 0 \) to the right of \( x = 2 \)
\( y < 0 \) to the left of \( x = 2 \)

Fig. 9.1 (b)
Linearization at the equilibrium point \((x, y) = (2, 4)\)

\[
\begin{bmatrix}
y' \\
\end{bmatrix} = \begin{bmatrix}
-2 & 1 \\
1 & 0 \\
\end{bmatrix} \begin{bmatrix}
y \\
\end{bmatrix} = A \begin{bmatrix}
y \\
\end{bmatrix}
\]

\((x, y) = (2, 4)\)

\[\begin{align*}
P_A(\lambda) &= \lambda^2 + 4\lambda - 1 = 0 \\
&\Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 4(-1)}}{2} \\
&\quad = -2 \pm \sqrt{5}
\end{align*}\]

\(\lambda_1 = -2 + \sqrt{5}, \quad \lambda_2 = -2 - \sqrt{5}\)

\[
A - \lambda I = \begin{bmatrix}
-2 - \sqrt{5} & 1 \\
1 & 2 - \sqrt{5} \\
\end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 + \sqrt{5} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 - \sqrt{5} \end{bmatrix}
\]
Consider the vector $\mathbf{v} = (a, b)$ in the direction field of a differential equation. The direction field shows the behavior of solutions near the equilibrium points. If $\mathbf{v}$ is in the direction of the equilibrium point, then the solution curves are straight lines parallel to $\mathbf{v}$. If $\mathbf{v}$ is in the opposite direction, the solution curves are straight lines parallel to $-\mathbf{v}$.

Example: For the system of differential equations

\[ \begin{align*}
\frac{dx}{dt} &= f(x, y) \\
\frac{dy}{dt} &= g(x, y)
\end{align*} \]

the equilibrium points are $(x_0, y_0)$, and the vector field near these points can be visualized using the direction field.

The nullclines are the curves where $f(x, y) = 0$ and $g(x, y) = 0$. These curves divide the phase plane into regions where the vector field points in different directions.

1. Nullclines
2. Direction field
3. Equilibrium points
4. Stability analysis
Figure 9.4: Nullclines and phase portrait for \( x, x' \).

Nullclines are defined by the equations \( x = 0 \) and \( x' = 0 \). The nullcline \( x = 0 \) is a horizontal line, and the nullcline \( x' = 0 \) is a vertical line. The phase portrait shows the direction of motion in the phase plane. The arrows indicate the direction in which solutions move.

Now let's look at the case \( a = 0 \). Here the system simplifies to:

\[
\dot{x} = ay, \quad \dot{y} = x
\]

which is equivalent to:

\[
\dot{x} = a, \quad \dot{y} = 1
\]

In this case, there are two solutions:

1. \( x = 0 \) and \( y = \frac{1}{a} \)
2. \( x = 1 \) and \( y = a \)

These solutions are unstable and unstable, respectively. The phase portrait shows the direction of motion in the phase plane. The arrows indicate the direction in which solutions move.

Figure 9.3: Nullclines and phase portrait for \( x, x' \).

Nullclines are defined by the equations \( x = 0 \) and \( x' = 0 \). The nullcline \( x = 0 \) is a horizontal line, and the nullcline \( x' = 0 \) is a vertical line. The phase portrait shows the direction of motion in the phase plane. The arrows indicate the direction in which solutions move.
Example

\[\dot{x} = x^2 - 1\]
\[y = -xy\]

**X-nullcline**

\[x^2 - 1 = 0 \Rightarrow x = \pm 1\]

**Y-nullcline**

\[x = 0 \text{ or } y = 0\]

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**Equilibrium points** (1, 0), (1, 0)

\[J = \frac{\partial}{\partial (x, y)} \begin{bmatrix} x^2 - 1 \\ -xy \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ -y & -x \end{bmatrix}\]

\[J(1, 0) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\]
\[\lambda_1 = 2, \quad \lambda_2 = -1, \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\]

\[J(-1, 0) = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\]
\[\lambda_1 = -2, \quad \lambda_2 = 1, \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\]
Stability of Equilibria

Theorem (Lyapunov Stability)

Let $\mathbf{x}^*$ be an equilibrium point for $\dot{\mathbf{x}} = f(\mathbf{x})$.

Let $L : \Omega \to \mathbb{R}$ be a differentiable function defined on an open set $\Omega$ containing $\mathbf{x}^*$.

Assume that

(a) $L(\mathbf{x}^*) = 0$ and $L(\mathbf{x}) > 0$ if $\mathbf{x} \neq \mathbf{x}^*$

(b) $L \leq 0$ in $\Omega - \mathbf{x}^*$

Then $\mathbf{x}^*$ is stable. Furthermore, if $L$ also satisfies

(c) $L < 0$ in $\Omega - \mathbf{x}^*$

Then $\mathbf{x}^*$ is asymptotically stable.

The concept of Lyapunov Stability is related to the concept of energy.
Example: Nonlinear Pendulum

\[ ml \frac{d^2 \theta}{dt^2} + bl \frac{d\theta}{dt} + mg \sin \theta = 0 \]

\[ m = l = g = 1 \]

\[ \dot{\theta} = v \]
\[ \dot{v} = -bv - \sin \theta \]

\[ G = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix} \]
\[ y = \begin{bmatrix} \theta \\ v \end{bmatrix} \]
Example: (The Nonlinear Pendulum) Consider a pendulum consisting of a massless, frictionless, rigid rod of length \( l \) hanging from a fixed point at the origin. The equation of motion for this pendulum is

\[ \ddot{\theta} + \frac{g}{l} \sin \theta = 0 \]

where \( \ddot{\theta} \) is the angular acceleration, \( g \) is the acceleration due to gravity, and \( l \) is the length of the rod. This equation is non-linear due to the \( \sin \theta \) term.

In Chapter 12, we will see that this equation can be linearized under certain conditions, leading to a simpler equation that can be solved analytically.

For now, let's focus on the nonlinear equation and explore its behavior. When \( \theta \) is small, the equation can be approximated by

\[ \ddot{\theta} + \frac{g}{l} \theta = 0 \]

which is equivalent to

\[ \ddot{\theta} + \frac{g}{l} \theta = 0 \]

This is a harmonic oscillator equation, which has solutions of the form

\[ \theta(t) = A \sin(\omega t + \phi) \]

where \( A \) is the amplitude, \( \omega = \sqrt{\frac{g}{l}} \) is the angular frequency, and \( \phi \) is the phase shift.

For \( \theta \) not small, the equation is non-linear, and the solutions are more complex. However, near the equilibrium points, the equation can be linearized, allowing for a more detailed analysis.
of a strict Lapunov function.

Figure 9.7. Solution curves through the level sets \( L_{-1} \).

\[
\begin{align*}
\sin z &= y \\
(1 + z + x)\dot{y} &= x \\
\dot{x} &= x
\end{align*}
\]

Example. Now consider the system:

However, after the 1st line, it is clear that the possibility of applying the linearity of the Lapunov function is limited to applying the linearity of the Lapunov function to the solution set of the Lapunov function. If \( \lambda \) is a quadratic function with \( ax^2 + bx + c \), then it is a quadratic function. For example, if \( \lambda \) is of the form \( (c) \), let \( c \) be the maximum of \( 0 \). Let \( a \) be the minimum of \( 0 \). Let \( b \) be the closed ball of \( \{ (x,y) \} \), and \( c \) be the minimum of \( 0 \).

Proof. Let \( b > 0 \) be small then the closed ball of \( \{ (x,y) \} \), and \( c \) be the minimum of \( 0 \).

We will return in the pendulum example for the case \( b < 0 \) later, but first we prove Lagrange's theorem.

Consider a fixed point and the associated system. If the system is simple to observe the behavior of the solution set of the Lapunov function, then it is possible to observe the behavior of the solution set of the Lapunov function. If \( \lambda \) is a quadratic function with \( ax^2 + bx + c \), then it is a quadratic function. For example, if \( \lambda \) is of the form \( (c) \), let \( c \) be the maximum of \( 0 \). Let \( a \) be the minimum of \( 0 \). Let \( b \) be the closed ball of \( \{ (x,y) \} \), and \( c \) be the minimum of \( 0 \).

Theorem. If \( x \) is the only possible initial point of the solution set, then \( \lambda = 0 \). The solution set corresponding to the pendulum example is the closed point of the equilibrium points at \( (x,y) = (0,0) \).

Figure 9.8. Phase portrait for the ideal pendulum.