Exam II Wed 11/1
HW #6 Chap 5 #2-5
(Due this Friday)
Read Chapter 5 Higher Dimensional Linear Algebra

Subspaces of a Vector Space

The span of vectors $v_1, \ldots, v_k$ is

$$\text{span} \{x_1 v_1 + \ldots + x_k v_k | x_i \in \mathbb{R}\}$$

Span $v_1, \ldots, v_k$

A subspace $S$ of $\mathbb{R}^n$ is a nonempty subset of $\mathbb{R}^n$ such that

$$x, y \in S \Rightarrow x + y \in S$$

Remark

A span of vectors is an example of a subspace.

If we have a subspace, then there are corresponding
vectors that span the subspace.

Examples 1.) let $e_1, e_2, e_3$ be the standard basis of $\mathbb{R}^3$. \[ \text{Span } e_1, e_2, e_3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R}^3 \right\} \]
\[ \text{A plane in } \mathbb{R}^3 \]

2.) let $\mathbb{P}^n = \text{real polynomials of order } n \text{ or less}$. Then $\mathbb{P}^n$ is $(n+1)$-dimensional vector space that is isomorphic to $\mathbb{R}^{n+1}$. The subset of even polynomials $E$ in $\mathbb{P}^n$ forms a subspace of $\mathbb{P}^n$.

Matrices $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ Any linear operator of $\mathbb{R}^n$ to $\mathbb{R}^n$ corresponds to an $n \times n$ matrix.

The Algebra of Matrices

\[ A + B = B + A, \quad A + (B + C) = (A + B) + C \]

\[ (AB)C = A(BC) \]

\[ A(B + C) = AB + AC, \quad (A + B)C = AC + BC \]

\[ k(AB) = (kA)B = A(kB) \text{ for any } k \in \mathbb{R}. \]
Matrix Inversion  An n x n matrix $A$ is invertible if there is a matrix $C$ such that $AC = CA = I$.

In this case, we call $C$ an inverse of $A$.

If a matrix $A$ has an inverse, then it is unique.

Proof  Suppose $B$ is another inverse of $A$.

Then $C = C(I) = C(AB) = (CA)B = IB = B$. □

Reduced Row Echelon Form

Elementary Row Operations:

1. Add $K$ times row $i$ of $A$ to row $j$ of $A$.
2. Interchange row $i$ and $j$.
3. Multiply row $i$ by $K \neq 0$.

These three elementary operations can convert any matrix $A$ to a reduced row echelon form.
Examples

Consider $\mathbb{R}^3$. The matrix associated with adding $2 \times$ the 1st row to the 3rd row is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

The matrix corresponding to interchanging the 1st and 3rd row is given by

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The matrix corresponding to multiplying the 2nd row by 2 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Important Result: Every matrix is the product of elementary matrices (i.e., matrices corresponding to elementary row operations).
Proposition Let $A$ be $n \times n$. The equation $A \mathbf{x} = \mathbf{v}$ has a unique solution for any $\mathbf{v} \in \mathbb{R}^n$ if and only if $A$ is invertible.

Proposition The matrix $A_{n \times n}$ is invertible if and only if the columns of $A$ form a set of linearly independent vectors.

Determinants How do we develop determinants rigorously?

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \]

\text{def} A ring is a set $K$ together with addition and multiplication satisfying

a) $K$ is a commutative group under addition

b) $x(yz) = (xy)z$ multiplication is associative

c) $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$ (Distributive law).
If $xy = yx$, we say the ring is commutative.

If there is an element $1$ such that $1x = x1 = x$, then the ring is said to have an identity.

An important example of a commutative ring with identity in $\mathbb{F}[x]$ (polynomials)

**Determinant Functions**

A function $D : M_n(F) \rightarrow F$ ($F = \mathbb{C}$ or $\mathbb{R}$ usually) is $n$-linear if $D(A)$ is linear in the $i$-th row with all other rows fixed.

**Examples**

Consider $2 \times 2$ matrices. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

The following are $2$-linear functions

1. $D_1(A) = a_{11}a_{22}$

2. $D_2(A) = -a_{12}a_{21}$

3. $D(A) = a_{11}a_{22} - a_{12}a_{21} = \text{det}(A)$
Let $D$ be an $n$-linear function. We say that $D$ is **alternating** if the following two conditions hold.

(a) $D(A) = 0$ whenever two rows of $A$ are equal.

(b) If $A'$ is a matrix obtained from $A$ by interchanging two rows of $A$, then $D(A') = -D(A)$.

Let $K$ be a commutative ring with identity and let $n$ be a positive integer. Suppose $D$ is a function from $n 	imes n$ matrices over $K$ into $K$. We say that $D$ is a **determinant function** if $D$ is $n$-linear, alternating, and $D(I) = 1$.

These properties are enough to uniquely define determinants.
Example 1.) Let \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \)

\[
D(A) = D(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) = D(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})
\]

\[
= f(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) \quad \text{E. } a_{21}, a_{22} \text{ held fixed}
\]

\[
= a_{11} D(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) + a_{12} D(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})
\]

\[
= a_{11} \left( a_{21} D(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) + a_{22} D(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \right)
\]

\[
+ a_{12} \left( a_{21} D(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) + a_{22} D(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \right)
\]

\[
= a_{11} \left( a_{21} D(I) + a_{22} D(I) \right)
\]

\[
+ a_{12} \left( a_{21} (-D(I)) + a_{22} (0) \right)
\]

\[
= a_{11} a_{22} - a_{12} a_{21}
\]
2.) Let \( A = [a_{ij}]_{3 \times 3} \). We get the standard determinant for a \( 3 \times 3 \) matrix.

**Useful Fact**

\[
A = \begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix}
\]

\[
\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}
\]