Last time we discussed the example
\[ x = \sin x = f(x). \]

Generally, it is difficult or impossible to find closed form solutions.

When \( f(x_0) = 0 \), we say that the system is at an equilibrium point.
If $x(0) = x_0$ where $0 < x_0 < 2\pi$ then $x(t) \to \pi$.

If $x(0) = x_0$ where $\pi < x_0 < 2\pi$ then $x(t) \to \pi$.

If $x(0) = x_0$ where $2\pi < x_0 < 3\pi$ then $x(t) \to 3\pi$.

Population Models

Last time we discussed exponential growth.

\[ \dot{x} = ax, \quad x(0) = x_0 \quad \Rightarrow \quad x(t) = e^{at} x_0 \]

\[ a = \text{birth rate - death rate} \]
A more accurate model in the logistics model

\[ \dot{x} = ax(1 - \frac{x}{N}) \quad x(0) = x_0 \quad N \text{ large} \]

For small populations, \( \dot{x} \approx ax \)
For larger populations, we have a smaller growth rate. Eventually \( x(t) \to N = \text{population capacity} \).

There are two ways to view this equation.

1) The growth rate \( ax \) decreases with population

\[ g = x \left( 1 - \frac{x}{N} \right) \leftrightarrow \text{simplest negative strictly decreasing function with value near 0 for small populations} \]

2) \( \dot{x} = ax - bx x^2 \)

\( \dot{x} \) \text{ natural growth rate due to conflict}

\[ G(t) = \text{"good people"} = px(t) \]
\[ B(t) = \text{"bad people"} = (1-p)x(t) \]

Bad interactions \( \propto G(t)B(t) \)

\[ = pG(t)B(t) = ppx(t)(1-p)x(t) \]
\[ = p(1-p)x^2(t) \]
\[ \dot{x} = \alpha x (1 - x) \rightarrow \frac{\dot{x}}{N} = a \frac{x}{N} (1 - \frac{x}{N}) \]

\[ \frac{d}{dt} \left( \frac{x}{N} \right) = a \left( \frac{x}{N} \right) \left[ 1 - \frac{x}{N} \right] \]

So replace \( \frac{x}{N} \) with \( x \) to get the above equation.

We will write \( \dot{x} = f_a(x) = a x (1-x) \)

\( f \) parametrized family of diff. eqns

Consider \( x = k (1-x) \).

\[ \int \frac{dx}{x(1-x)} = \int dt \]

\[ \int \left[ \frac{1}{x} + \frac{1}{1-x} \right] dx = dt \]

\[ \ln x - \ln (1-x) = t + C \]

\[ \ln \left( \frac{x}{1-x} \right) = t + C \]
\[
\frac{X}{1-x} = Ke^t
\]

\[\Rightarrow X(t) = \frac{Ke^t}{1+Ke^t}\]

\[
\text{Tricks of the Trade}
\]

\[
y = \frac{ax+b}{cx+d} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}
\]

\[
\begin{bmatrix} x \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix}}{ad-bc}
\]

\[
\frac{X}{1} = \frac{dy-b}{-cy+a}
\]

\[\text{Note that the determinant is lost due to the division}\]

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

In our example above, we have

\[
\frac{X}{1-x} \rightarrow \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \rightarrow y = Ke^t
\]

\[
x(t) = \frac{Ke^t}{Ke^t+1}
\]
Another quick trick is \( \frac{1}{dx} \left( \frac{ax+b}{cx+d} \right) = \frac{ad-bc}{(cx+d)^2} \), numerator is det of \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).}

The general solution is

\[
X(t) = \frac{NKe^at}{1 + Ke^at}
\]

We obtain \( K \) from initial conditions. Back to

\[
\dot{X} = X(1-X) \rightarrow \frac{X}{1-X} = Ke^at
\]

For \( X(0) = X_0 \), we have \( K = \frac{X_0}{1-X_0} \)

and \( X(t) = \frac{X_0}{1-X_0} e^{at} \)

\[
= \frac{X_0 e^{at}}{1 - X_0 + X_0 e^{at}}
\]

The more general solution

\[
X(t) = \frac{N X_0 e^{a(t-t_0)}}{1 - X_0 + e^{a(t-t_0)} X_0} \quad \text{where} \quad X(t_0) = X_0
\]
Note that \( x(t) \to N \) as \( t \to 0^+ \)

Let's now apply the geometric method

\[ f(x) = x(1-x) \]
Example 1) \( \dot{x} = x - x^3 = x(1-x^2) = x(1-x)(1+x) \)

\[ = -(x-1)x(x+1) = f(x) \]

2) \( \dot{x} = x^2 \)

Exercise Show that this has a finite escape time.

1.3 Constant Harvesting and Bifurcations

\[ \dot{x} = x(1-x) - h \]

See Fig. 1.6 and 1.7.